Math 1433

22 January 2024

Warm-up:

1) Factor the number 75.

2) List the factors of 75.

(several answer)

1, 3, 5, 15, 25, 75

Last week

Polynomials

A **polynomial** in the variable x is a function that *can* be described by an expression of the form

$$😀x^n + 🧐x^{n-1} + \cdots + 😂x^2 + 🥺x + 😐,$$

where $n \geq 0$ is an integer and the emoji are real or complex numbers (called the **coefficients**).

A **real polynomial** is one where every coefficient is a real number.

A **complex polynomial** is one where every coefficient is complex.

- Real numbers are complex numbers ($a + 0i$), so every real polynomial is also a complex polynomial.

Term, Degree, Root

Last week

The **degree** of a polynomial is the highest power of the variable that appears in the polynomial.

- Degree 0 example: 9 "constant"
- Degree 2 example: $z^2 + z + 5 + 6i$ "quadratic"
- Degree 3 example: $x^3 + \sqrt{7}x^2 - 8x + 2$ "cubic"
- Degree 19 example: $\frac{1}{3}x^{19} - 4x + 1$ "degree 19"

A **zero** of the polynomial f is a number c for which $f(c) = 0$. This is also called a **root** of the polynomial.

Task: Find all roots of $x^2 - 13x + 12$.

Use the "Quadratic Formula" to solve $ax^2 + bx + c = 0$.

$\Delta = b^2 - 4ac$

$\Delta = (-13)^2 - 4(1)(12) = 121 \quad \rightarrow \quad \sqrt{\Delta} = 11$

$$x_1 = \frac{-(-13)+11}{2(1)} = \frac{24}{2} = \boxed{12} \quad \text{and} \quad x_2 = \frac{-(-13)-11}{2(1)} = \boxed{1}$$

Task: Find all roots of $x^3 - 13x + 12$.

(This is harder. We need some new tools.)

Factoring

Natural numbers can be “factored” (re-written as a product of smaller numbers).

- Example: $198 = 6 \cdot 33$

If $a = b \cdot c$, we say that b is a **factor** of a.

A natural number other than 1 that cannot be factored is called a **prime** number. The first several primes are $2, 3, 5, 7, 11, 13, \ldots$

We can *uniquely* factor a natural number as a product of primes.

- Example: $198 = 2 \cdot 3^2 \cdot 11$

(If we expand from naturals to integers, we might need to include -1.)

- Example: $-1625 = -1 \cdot 5^3 \cdot 13$

Factoring

Polynomials can also be factored. Examples:

- $x^2+8x = x(x+8)$
- $x^2+\frac{1}{2}x = x(x+\frac{1}{2})$
- $x^2 - 5 = (x+\sqrt{5})(x-\sqrt{5})$
- $x^3-12x^2+41x-42 = (x^2-5x+6)(x-7)$
- $x^3-11x^2+34x-42 = (x^2-4x+6)(x-7)$

If $f(x) = g(x)\cdot h(x)$, we say that $g(x)$ is a **factor** of $f(x)$.

Factoring

Polynomials can also be factored.

If $f(x) = g(x) \cdot h(x)$, we say that $g(x)$ is a **factor** of $f(x)$.

The Factor Theorem

Let $f(x)$ be a polynomial. r is a zero of f if and only if $(x - r)$ is a factor of f.

Question: Why is this useful?

Answer: If we find one zero of $f(x)$ (call it r) then the other zeros of $f(x)$ will be zeros of $g(x) = \frac{f(x)}{x - r}$. Note that g has a lower degree than f.

Example: Find all roots of $x^3 - 13x + 12$, given that 3 is a root.

Slow method: lots of algebra

Medium-fast method: "long division"

Fast method: "synthetic division"

Answer: -4, 1, 3

Finding roots by hand

Rational Root Theorem

If $Ax^n + \cdots + Yx + Z$ has integers for all coefficients, and $\pm\frac{p}{q}$ is a root of this polynomial, then p is a factor of Z and q is a factor of A.

With $A = 1$ we get a simpler version: "If $x^n + \cdots + Yx + Z$ has integer coefficients and $\pm p$ is a root of this polynomial, then p is a factor of Z."

Example: Find all roots of $x^3 - 13x + 12$.

Factors of 12 are 1, 2, 3, 4, 6, 12.

So check these numbers: 1, -1, 2, -2, 3, -3, 4, -4, 6, -6, 12, -12.

Once we find a good root, this becomes the previous task.

Task 1: find the roots of $x^2 + 10x + 14$.

- Answer: $-5 + \sqrt{11}$ and $-5 - \sqrt{11}$

Task 2: <u>factor</u> $x^2 + 10x + 14$.

- Answer: $(x + 5 - \sqrt{11})(x + 5 + \sqrt{11})$

Do this now: <u>factor</u> $3x^2 + 5x + 7$.

$$\left(x - \frac{-5+\sqrt{59}i}{6}\right)\left(x - \frac{-5-\sqrt{59}i}{6}\right)$$

Suppose $f(r) = 0$ and the coefficients of f are all real numbers. Using some algebra rules (e.g., $(\bar{z})^2 = \overline{z^2}$), we show that $f(\bar{r}) = \overline{f(r)} = \bar{0} = 0$.

Conjugate Pairs Theorem

If $a + bi$ is a root of a *real* polynomial, then $a - bi$ is also a root of that polynomial.

Example: $4 + 3i$ is one of the zeros of $f(z) = 2z^3 - 19z^2 + 74z - 75$. Knowing this, find all the zeros of f without a calculator.

Answer: 4+3i, 4-3i, 3/2

Last week

What numbers z satisfy $z^2 = 1$?

Answer: 1, -1

What are all the complex numbers z that satisfy $z^4 = 1$?

Note that $z^2 = 1$ or $z^2 = -1$.

Answer: 1, -1, i, -i

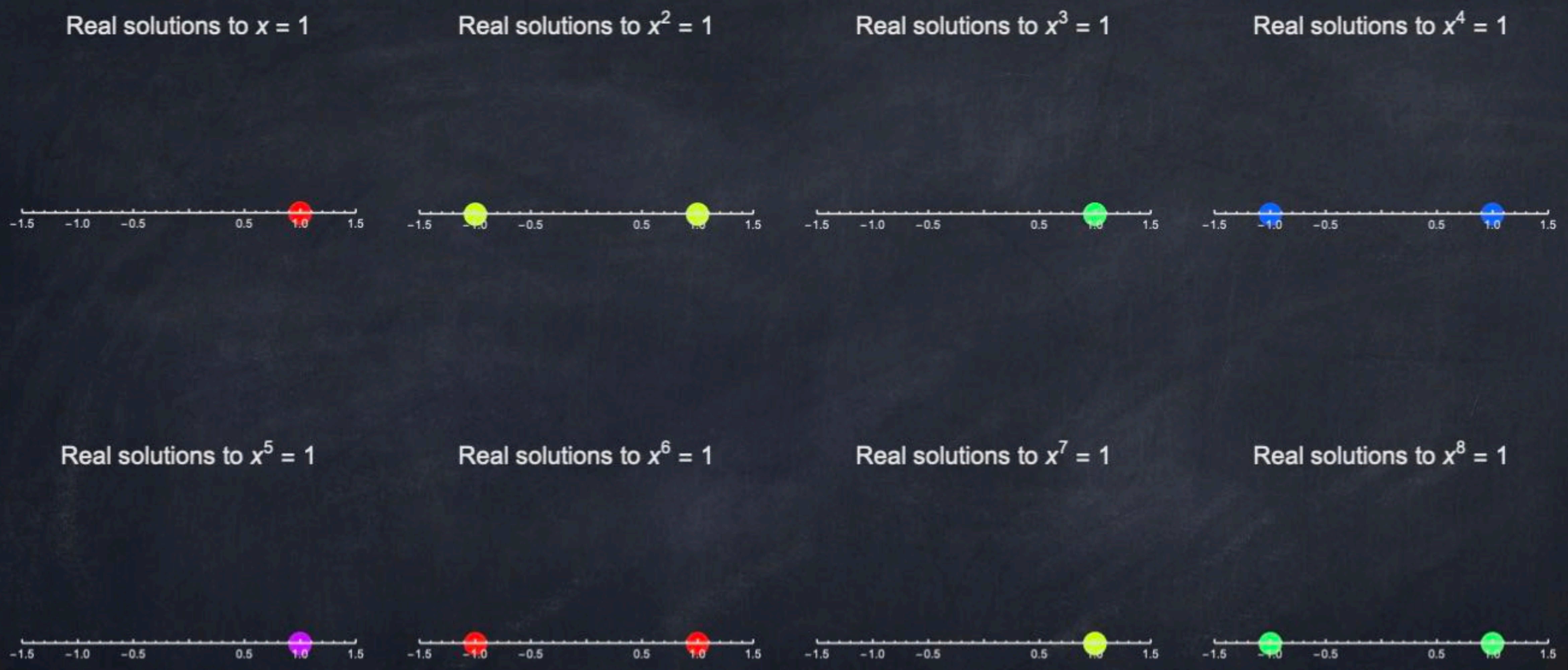
Roots of unity

For any natural number n, the solutions to $z^n = 1$ are exactly

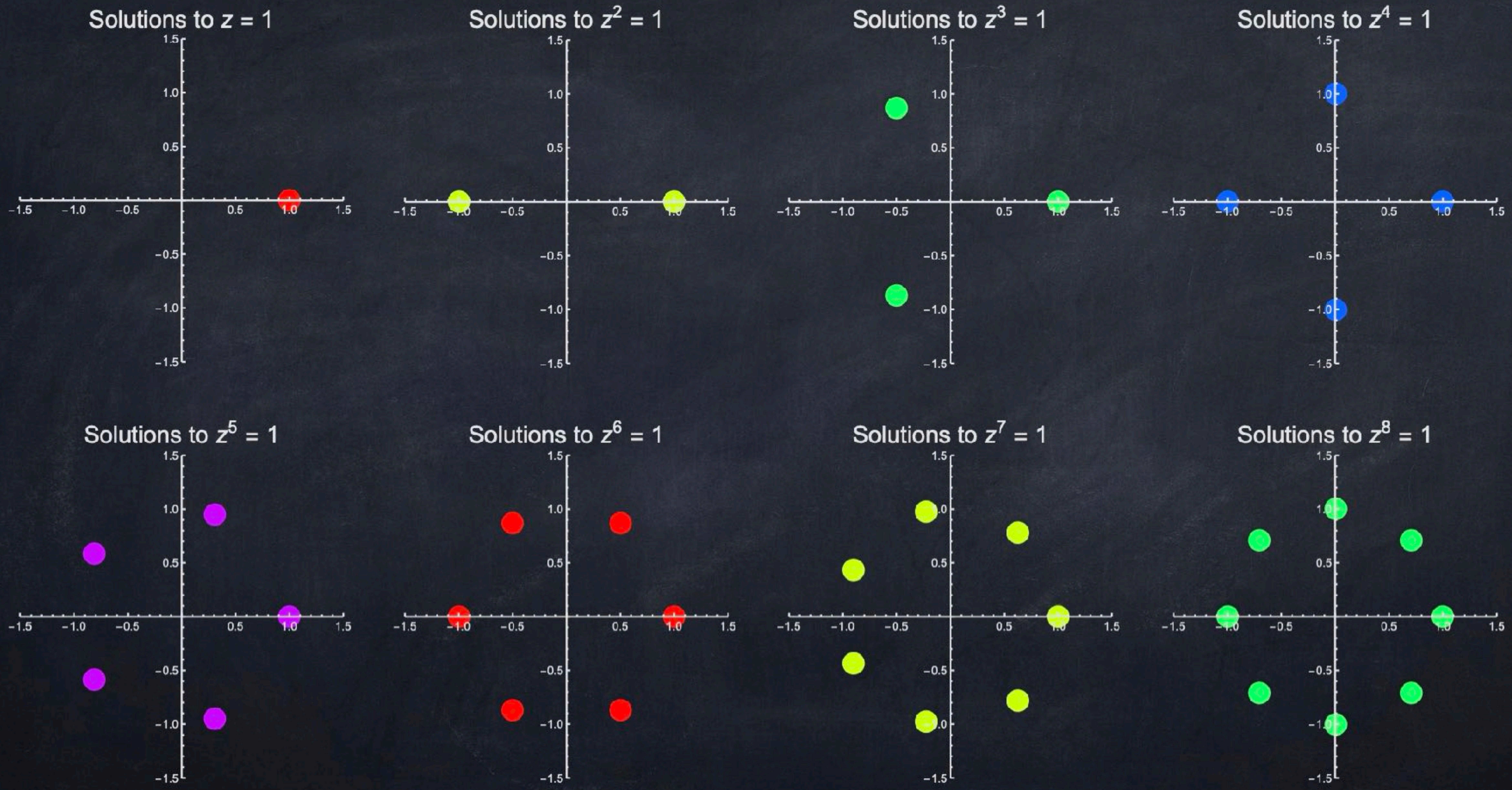
- $z = e^{(2\pi/n)i}$ ← Call this ω.
- $z = e^{2\cdot(2\pi/n)i} = \omega^2$
- $z = e^{3\cdot(2\pi/n)i} = \omega^3$

  $\vdots$
- $z = e^{(n-1)\cdot(2\pi/n)i} = \omega^{n-1}$
- $z = e^{n\cdot(2\pi/n)i} = \omega^n = 1.$

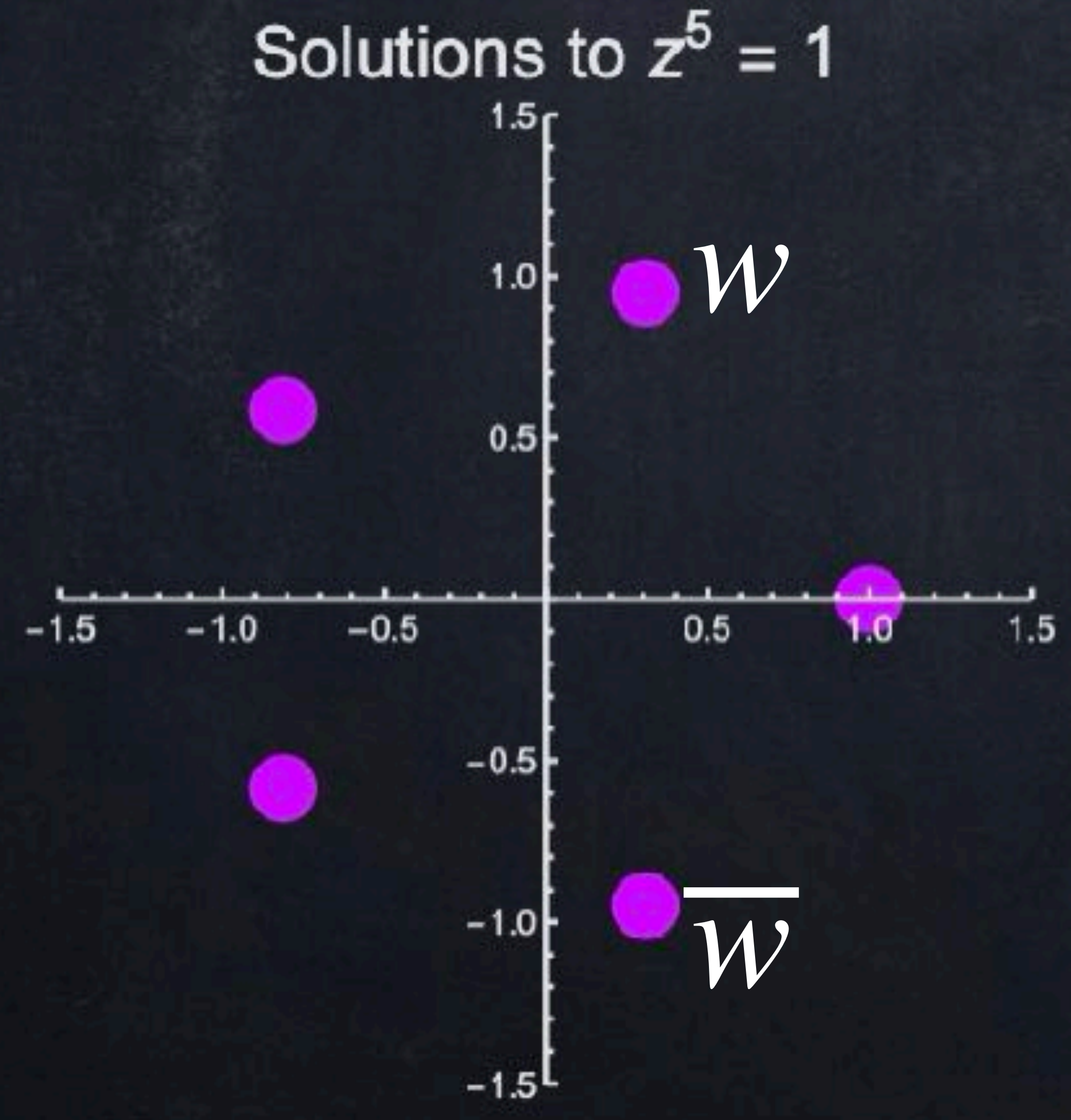
These are called the ***n*^th^ roots of unity**.



For real $x^n = 1$, it is different when n is even or odd.
But both kinds of pictures are boring.


Solutions to $z = 1$
Solutions to $z^2 = 1$
Solutions to $z^3 = 1$
Solutions to $z^4 = 1$
Solutions to $z^5 = 1$
Solutions to $z^6 = 1$
Solutions to $z^7 = 1$
Solutions to $z^8 = 1$

The application of complex numbers to geometry—of which these pictures is one nice example—was not understood by Cardano or the others who first used $\sqrt{-1}$ in the 1500s to solve equation. The increased popularity of complex numbers in the 1800s is partly because mathematicians realized the uses of complex numbers beyond just solving equations.



If you look closely at any one of the previous pictures, you can see that for every dot, there is also a dot at the conjugate.

Irreducible factors

A polynomial that can be factored as a product of non-constant polynomials is called **reducible**.

Note: $2x+10$ is irreducible even though $2x+10 = 2(x+5)$.

A polynomial that is not reducible is called **irreducible**.

- *Question:* How can we tell when a polynomial is irreducible?

$$x^3-12x^2+41x-42 = (x^2-5x+6)(x-7) = (x-2)(x-3)(x-7)$$

$$x^3-11x^2+34x-42 = (x^2-4x+6)(x-7)$$

Irreducible factors

Question: How can we tell when a polynomial is irreducible?

- Any linear polynomial must be irreducible.
- What about quadratic polynomials?
  - The roots of ax^2+bx+c are $\dfrac{-b \pm \sqrt{D}}{2a}$, where $\Delta = b^2 - 4ac$.

    (The number Δ is called the discriminant.)
  - With complex numbers we can always factor quadratics using

$$ax^2+bx+c = \left(x - \frac{-b+\sqrt{\Delta}}{2a}\right)\left(x - \frac{-b-\sqrt{\Delta}}{2a}\right),$$

    but for real numbers we need $\Delta \geq 0$ in order to use $\sqrt{\Delta}$.

Irreducible factors

Question: How can we tell when a polynomial is irreducible?

Answer: It depends on whether you allow complex numbers.

- An irreducible *complex polynomial* is linear.
- An irreducible *real polynomial* is either linear or is quadratic with $\Delta < 0$.

Remember, the name “real polynomial” refers to the coefficients. A real polynomial can still have complex roots.

Task: Write $2z^3 - 19z^2 + 74z - 75$ as a product of irreducible complex polynomials.

- Reminder from earlier: the roots are $4 + 3i$, $4 - 3i$, $3/2$

The answer to this kind of task will always look like $(z-c_1)(z-c_2)\cdots(z-c_n)$. For this function, it is

$$(z - (4+3i))(z - (4-3i))(2z - 3)$$

Task: Write $2x^3 - 19x^2 + 74x - 75$ as a product of irreducible real polynomials.

This might be $(x-c_1)\cdots(x-c_n)$, but it might have quadratic factors.

$$(x^2 - 8x + 25)(2x - 3)$$

Irreducible factors

The Fundamental Theorem of Algebra (version 2)

A complex polynomial of degree n can be factored into exactly n irreducible complex factors.

(We saw version 1 last week.)

This requires complex numbers. For example:

$$x^4 + x^3 - 21x^2 + 9x - 270 \quad = \quad (x-5)(x+6)(x^2+9)$$

but

1 2 3 4

$$z^4 + z^3 - 21z^2 + 9z - 270 \quad = \quad (z-5)(z+6)(z+3i)(z-3i)$$

Repeated roots

We already know that a number r is a **root** of f (also, r is a **zero** of f) if and only if

$$f(x) = (x - r)h(x).$$

for some polynomial h.

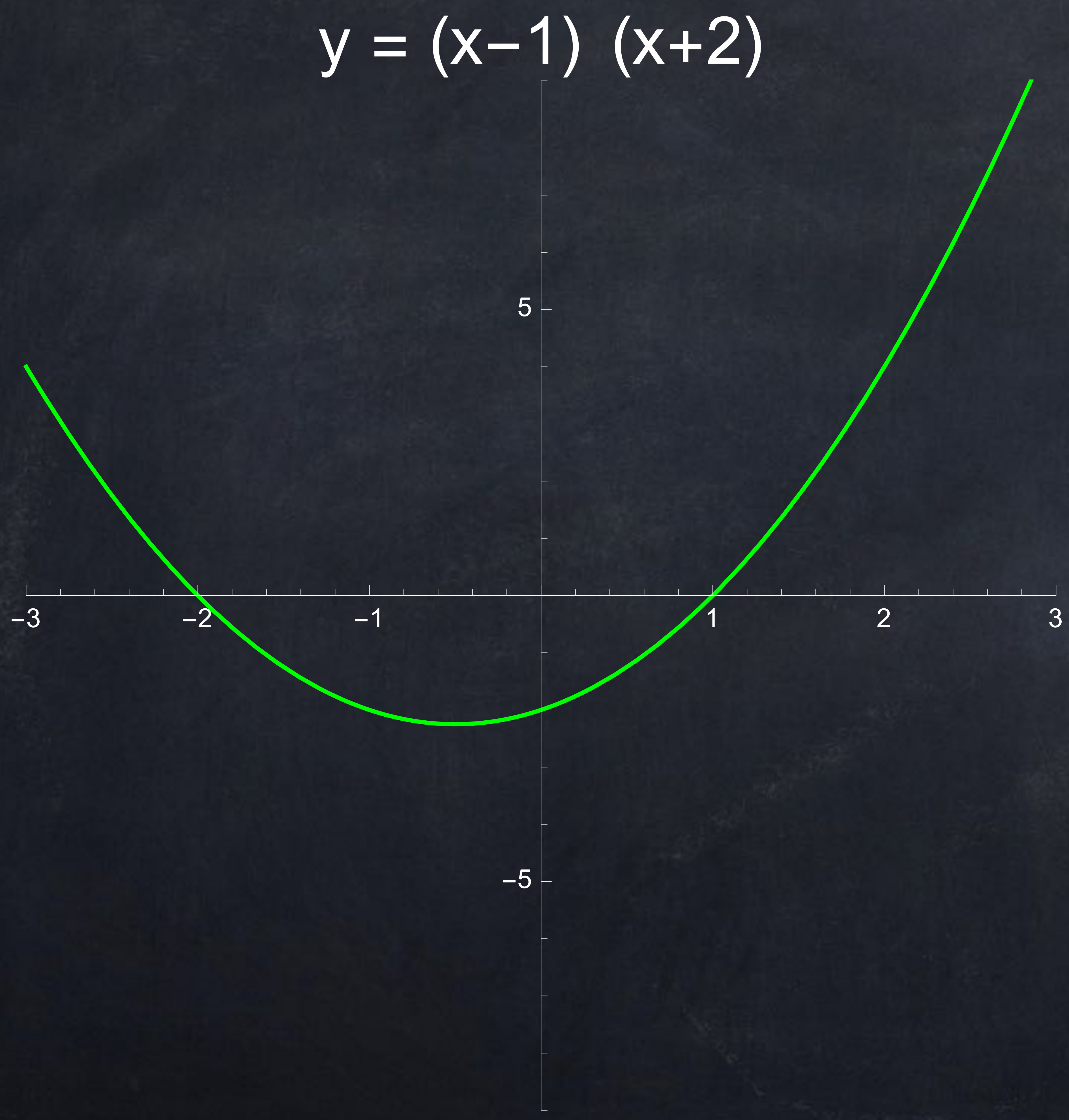
The **multiplicity** of the root r is the highest number m such that

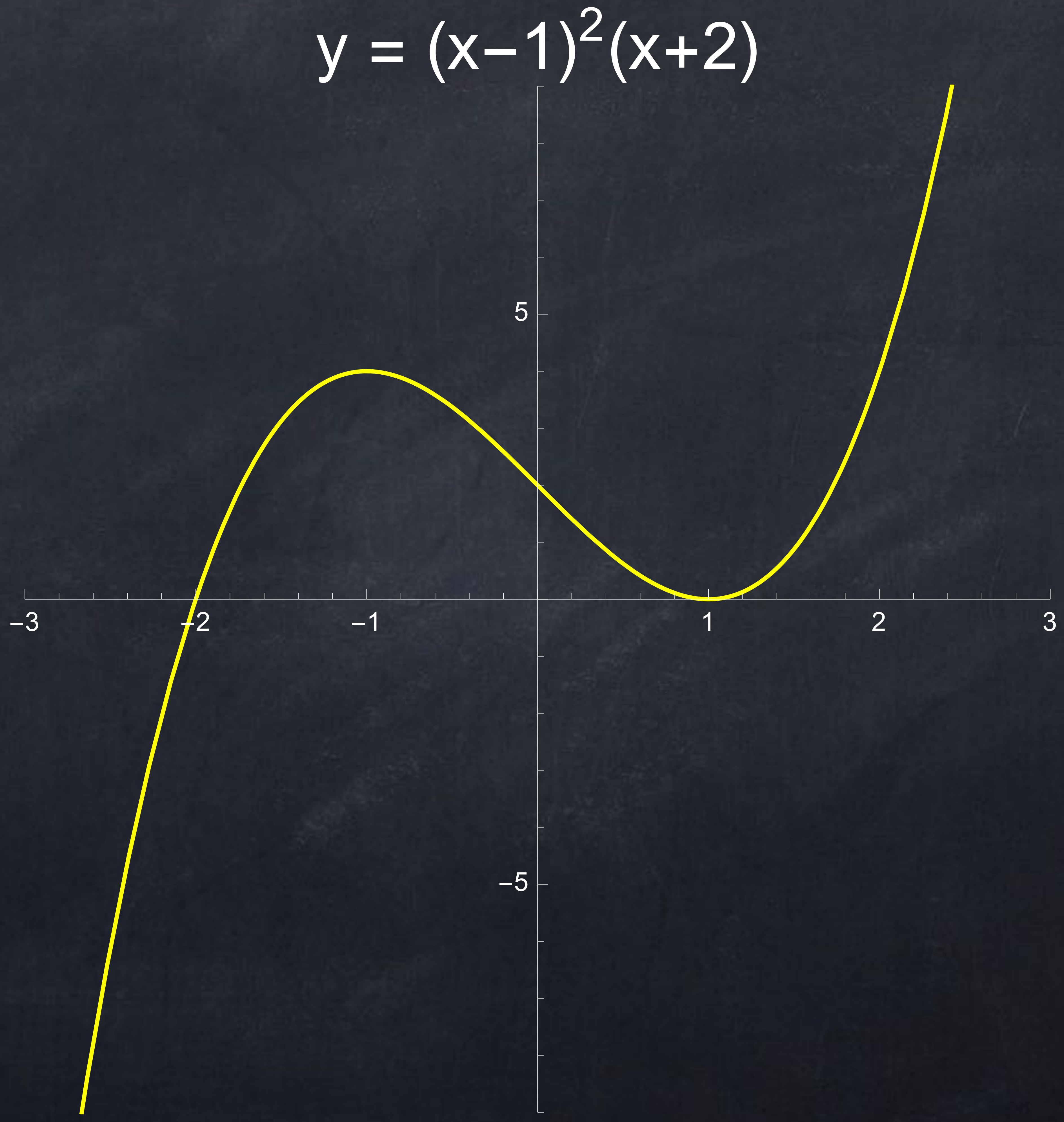
$$f(x) = (x - r)^m g(x)$$

for some polynomial g.

- If $m > 1$ we say that r is a **repeated root** of f.

Roots of a polynomial





The polynomial $x^4 + 4x^3 - 18x^2 + 20x - 7$ has $x = 1$ as a root. What is the multiplicity of this root?

Fast version OF ALL STEPS:

	1	4	-18	20	-7
1		1	5	-13	7
	1	5	-13	7	0
1		1	6	-7	
	1	6	-7	0	
1		1	7		
	1	7	0		
1		1			
	1	8			

Answer: 3

8 ← not divisible by x-1 anymore → stop

Fundamental Theorem of Algebra (version 3)

A polynomial of degree n has exactly n complex zeros, counted with multiplicities.

Example: $f(z) = z^7 + 11z^6 + 41z^5 + 43z^4 - 69z^3 - 135z^2 + 27z + 81$.

- The only numbers for which $f(z) = 0$ are -3, -1, and 1.
- Since $f(z) = (z+3)^4(z+1)(z-1)^2$, we can think of the zeros of f as

$$-3,\ -3,\ -3,\ -3,\ -1,\ 1,\ 1.$$

How many roots does $97344x^2 - 327600x + 275625$ have?

- Distinct real roots: Is $D = b^2 - 4ac$ positive, zero, or negative? 🤔

  ↓ two roots ↓ one root ↓ no roots

- Distinct complex roots: Is $D = b^2 - 4ac$ is zero or not? 🤔

  ↓ one root ↓ two roots

- Complex roots with multiplicities: 2. 😀

The "best" way to write a polynomial depends on your goal.

- $x^2 + 3x + 2$ is standard form. It is good for testing whether two polynomials are exactly equal.
- $(x + 3)x + 2$ is good for plugging in x-values (only one multiplication).
- $\left(x+\frac{3}{2}\right)^2-\frac{1}{4}$ is good for graphing (vertex is at $\left(\frac{-3}{2}, \frac{-1}{4}\right)$).
- $(x + 1)(x + 2)$ is good for finding zeros.

There are also multiple useful ways to write a **rational function**, which is one polynomial divided by another.

Partial fractions

$\frac{4x+1}{x^2+x}$ is equal to $\frac{3}{x+1} + \frac{1}{x}$.

- If you start with $\frac{3}{x+1} + \frac{1}{x}$ and want to re-write this as one fraction, you would first have to make the denominators equal: $\frac{3x}{x(x+1)} + \frac{x+1}{x(x+1)}$.

- The idea of "partial fraction decomposition" is to start with $\frac{f(x)}{g(x)}$, where f and g are polynomials, and write it as a sum in a certain way.

A **partial fraction** is a rational function where the denominator is a power of an irreducible (real) polynomial and the numerator has a lower degree than the irreducible polynomial.

Examples: $\frac{9}{3x-5}$ $\qquad$ $\frac{9}{(3x-5)^2}$ $\qquad$ $\frac{4x+2}{3x^2+5x+7}$

You do *not* need to know the exact definition above. But you need to know the steps to re-write $\frac{f(x)}{g(x)}$ as a sum of partial fractions:

1. Factor $g(x)$ into irreducible factors.
2. Each irreducible factor of $g(x)$ will be the denominator of a partial fr.
3. Use algebra to find the numerators.

Example: Write $\dfrac{13x+9}{x^2+3x-10}$ as a sum of partial fractions.

Since $x^2+3x-10 = (x-2)(x+5)$, we are looking for

$$\frac{13x+9}{(x-2)(x+5)} = \frac{A}{x-2} + \frac{B}{x+5}$$

$$13x+9 = A(x+5) + B(x-2)$$

After more algebra … $A = 5$ and $B = 8$.

Final answer: $\dfrac{5}{x-2} + \dfrac{8}{x+5}$

Partial fractions setup

If $g(x) = (x-a)(x-b)\cdots$ with distinct linear factors, then writing $\frac{f(x)}{g(x)}$ as a sum of partial fractions is just like our previous example.

$$\frac{f(x)}{2x^4 + 5x^3 - 60x^2 + 25x + 28} = \frac{f(x)}{(x-4)(x-1)(2x+1)(x+7)}$$

$$= \frac{A}{x-4} + \frac{B}{x-1} + \frac{C}{2x+1} + \frac{D}{x+7}$$

$$\text{for some } A, B, C, D$$

Partial fractions setup

If g has an irreducible quadratic (degree 2) factor, we need a linear (degree 1) *numerator* for that fraction:

$$\frac{f(x)}{x^3 + x^2 - 8x + 238} = \frac{f(x)}{(x+7)(x^2 - 6x + 34)} = \frac{A}{x+7} + \frac{Bx + C}{x^2 - 6x + 34}$$

for some A, B, C

If g has repeated zeros, we need a partial fr. for each power:

$$\frac{f(x)}{(x-8)^3(x+1)} = \frac{A}{x-r} + \frac{B}{(x-r)^2} + \frac{C}{(x-r)^3} + \frac{D}{x+1}$$